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TO‘RTBURCHAK SOHADA TELEGRAF TENGLAMA UCHUN DIRIXLE MASALASI

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Annotatsiya: *Mazkur ishda to‘g‘ri to‘rtburchak sohada telegraf tenglamasi uchun birinchi chegaraviy masala o‘rganilgan. Masala yechimi yagonaligining mezoni ishlab chiqilgan. Masala yechimi ortogonal qatorlar yig‘indisi shaklida qurilgan bo‘lib, qatorlar yaqinlashishini asoslash jarayonida kichik maxrajlar muammosi yuzaga kelgan. Tegishli asimptotik formulalardan foydalanib, maxrajning noldan farqli ekanligini ko‘rsatuvchi baholar topilgan. Regulyar yechimlar sinfida yechimning mavjudli ko‘rsatib o‘tilgan.*

Kalit so‘zlar: *telegraf tenglamasi, Dirixle masalasi, spektral usul, masala yechimi yagonaligi, masala yechim mavjudligi, kichik maxrajlar muammosi*

I.Kirish. Masalaning qo‘yilishi

Aytaylik bizga $D=\{(x,y):0<x<l,\,0<y<T\}$ soha berilgan bo‘lsin, bu yerda l,T - musbat o‘zgarmaslar. Ushbu sohada quyidagi

$$Lu\equiv u_{xx}-u_{yy}-b^2u=0,\quad b=const>0, \tag{1}$$

(1) telegraf tenglamasini ko‘rib chiqamiz.

D sohada (1) tenglama uchun birinchi chegaraviy masalani qo‘yamiz va tadqiq qilamiz.

Dirixle masalasi. D sohada quidagi

$$u(x,y)\in C(\overline{D})\cap C^2(D); \tag{2}$$

$$Lu=0,\quad (x,y)\in D; \tag{3}$$

$$u(0,y)=u(l,y)=0,\quad 0\le y\le T; \tag{4}$$

$$u(x,0)=\varphi(x),\; u(x,T)=\psi(x),\; 0\le x\le l, \tag{5}$$

shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani topilsin, bu yerda $\varphi(x),\psi(x)$-yetarlicha silliq funksiyalar va bu funksiyalar uchun quidagi

$$\varphi(0)=\varphi(l)=0\;,\psi(0)=\psi(l)=0.$$

kelishuv shartlari o‘rinli.

Ma’lumki, gipepbolik tipdagi tenglamalar uchun Dirixle masalasi har doim ham korrekt qo‘yilgan masala hisoblanmaydi. Dirixle masalasini tor tebranish tenglamasi uchun o‘rganishga oid dastlabki ishlar J. Adamar [1], A. Guber [2] va D. Manjeron [3] tomonidan amalga oshirilgan. Keyinchalik ushbu masala ko‘plab olimlarning e’tiborini tortgan [4]–[15]. Mazkur mavzuga bag‘ishlangan ishlardagi natijalarning tahlilini V.I. Arnold [12],

[13], Yu.M. Berezanskiy [16], V.G. Mazi va T.O. Shaposhnikova [17] asarlarida topish mumkin.

Endi esa Dirixle masalasi faqatgina to‘g‘ri to‘rtburchak sohada o‘rganilgan ishlarni keltiramiz. P. Burjen va R. Daffinlar [4], [5] $u_{xx} - u_{yy} = 0$ tenglama uchun chegaraviy masalani $0 \le x \le a,\ 0 \le y \le b$ to‘rtburchakda o‘rgandilar. Laplas almashtirishdan foydalangan holda, P. Burjen va R. Daffinlar tomonidan qiyidagitasdiq isbotlangan: agar tomonlar nisbati b/a irratsional bo‘lsa, uzluksiz differensiallanuvchi funksiyalar sinfida yechimning yagona ekanligi haqidagi teorema to‘g‘ri bo‘ladi.

K.B. Sabitov [15] tomonidan tomonlar nisbati T/l irratsional bo‘lishi, to‘rtburchak sohaga mos keluvchi tor tebranish tenglamasi uchun Dirixle masalasi yechimining yagona ekanligi uchun zarur va yetarli shartlar topilgan. Agar T/l algebraik irratsional son bo‘lsa, u holda Dirixle masalasining yechimi qator yig‘indisi shaklida qurilgan va yechimning uzluksiz funksiyalar sinfda yaqinlashuvchi ekanligiga asoslangan.

B.V. Kapitonov [14] ishida Dirixle masalasi asosida (1)-tenglama uchun aylanuvchi suyuqlikning kichik tebranishlari tenglamalariga oid ikki o‘lchovli model masalalari o‘rganilgan. V.I. Arnold ([18], 131–136-betlar) raketaning ingichka devorli idishlaridagi suyuqlik tebranishlarining rezonanslari bilan bog‘liq beqaror tebranishlar muammolari va giperbolik tenglamalar uchun Dirixle masalasi o‘rtasidagi bog‘liqlikni ta’kidlab o‘tgan.

Ushbu ishda [15], [19] ishlarining g‘oyalaridan foydalanib, biz (1)-tenglama uchun (2)–(5) masala yechimining yagona bo‘lishi, mavjudlik teoremasi va yechimning regulyar ekanligini isbotlaymiz.

II.Dirixle masalasini yechimining yagonaligi

Dirixle masalasida o‘zgaruvchilarni $u(x, y) = X(x)Y(y)$ formula bo‘yicha ajiratamiz va x o‘zgaruvchi bo‘yicha quyidagi spektral masalani hosil qilamiz:

$$X''(x) + \mu^2 X(x) = 0, \qquad 0 < x < l,$$

(6)

$$X(0) = X(l) = 0,$$

(7)

bu yerda μ - ajratish o‘zgarmasi. {(6), (7)} ko‘rinishdagi spektral masala ning xos qiymatlari va ularga mos xos funksiyalari quyidagi ko‘rinishda topiladi:

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \mu_n x, \qquad \mu_n = \frac{\pi n}{l},\ n = 1, 2, \ldots\ldots$$

(8)

$\{X_n(x)\}_{n=1}^{\infty}$ funksiyalar sistemasi $L_2[0, l]$ fazoda to‘la ortogonallangan bazis tashkil qiladi.

Aytaylik $u(x, y)$ masalaning yechimi bo‘lsin. [20], [15] ga asosan quyidagi yordamchi funksiyani qaraymiz:

$$u_n(y)=\int_0^l u(x,y)X_n(x)dx,\quad 0\le y\le T,\quad n=1,2,... \quad . \tag{9}$$

Keyinchalik biz xususiy hosila $u_x(x,y)$ quyidagi

$$\lim_{x\to 0+} u_x(x,y)X_n(x)=\lim_{x\to l-} u_x(x,y)X_n(x)=0,\quad 0\le y\le T.$$

(10)

(10) shartlarni qondiradi deb faraz qilamiz.

(9) ga asosan, quyidagi ifodani kiritamiz:

$$u_{\varepsilon,n}(y)=\int_{\varepsilon}^{l-\varepsilon} u(x,y)X_n(x)dx,\quad n=1,2...,\tag{11}$$

bu yerda $\varepsilon>0$ - yetarlicha kichik son. (11) ni ikki marta differensiallaymiz va (1) tenglamani hisobga olgan holda

$$u''_{\varepsilon,n}(y)=\int_{\varepsilon}^{l-\varepsilon} u_{yy}(x,y)X_n(x)dx=\int_{\varepsilon}^{l-\varepsilon} u_{xx}(x,y)X_n(x)dx-b^2\int_{\varepsilon}^{l-\varepsilon} u(x,y)X_n(x)dx$$ tenglikni hosil qilamiz.

Oxirgi ifodaning o‘ng tomonining birinchi integralida ikki marta bo‘laklash amalini bajarib, so‘ngra $\varepsilon\to 0$ chegarasiga o‘tayotganda (4), (7) va (10) shartlarni hisobga olgan holda quyidagi differensial tenglamani hosil qilamiz:

$$u''_n(y)+\lambda^2 u_n(y)=0,\quad \lambda^2=\mu^2+b^2.$$

(12)

(12) tenglamaning umumiy yechimi quidagi formula bilan aniqlanadi:

$$u_n(y)=a_n\cos\lambda_n y+b_n\sin\lambda_n y\ ,$$

(13)

bu yerda a_n va b_n -ixtiyoriy o‘zgarmas sonlar. Noma’lum a_n va b_n konstantalarini topish uchun (5) chegaraviy shartlardan va (9) formuladan foydalanamiz:

$$u_n(0)=\int_0^l u(x,0)X_n(x)dx=\int_0^l \varphi(x)X_n(x)dx=\varphi_n,$$

(14)

$$u_n(T)=\int_0^l u(x,T)X_n(x)dx=\int_0^l \psi(x)X_n(x)dx=\psi_n.$$

(15)

(13) funksiyalarni (14) va (15) chegaraviy shartlarni qanoatlantirib, barcha $n\in N$ uchun quyidagi

$$\delta(n)=\sin\lambda_n T\ne 0$$

(16)

shart bajarilishini qabul qilamiz va koeffitsientlarni quyidagi ko‘rinishda topamiz:

$$a_n = \varphi_n, \quad b_n = \frac{\psi_n - \varphi_n \cos\lambda_n T}{\sin\lambda_n T}. \qquad (17)$$

(17) ni (13) formulaga qo‘yib, funksiyalarning oxirgi ko‘rinishda topamiz.

$$u_n(y) = \varphi_n \cos\lambda_n y + \frac{\psi_n - \varphi_n \cos\lambda_n T}{\sin\lambda_n T}\sin\lambda_n y \qquad (18)$$

1-teorema Agar (2)–(5) masalaning (10) shartlarni qanoatlantiruvchi yechimi mavjud bo‘lsa, u holda uning yagona bo‘lishi uchun barcha $n \in N$ lar uchun (16) shart bajarilishi zarur va yetarlidir.

Isbot. Zarurlik. Faraz qilaylik, (2)–(5) masalaning yagona yechimi mavjud. Barcha $n \in N$ uchun (16) shart bajarilishini ko‘rsatamiz. Aytaylik , ma’lum bir p qiymati uchun $\delta(p) = \sin(\lambda_n T) = 0$ holatda $\varphi(x) = \psi(x) \equiv 0$ bo‘lganda, (2)–(5) masalasining noldan farqli yechimi ya’ni nolga teng bo‘lmagan yechim quyidagi ko‘rinishda ga ega bo‘ladi:

$$u_p(x, y) = \sin\mu_p x \sin\lambda_n y \qquad (19)$$

Yetarlilik. Faraz qilaylik, $[0,l]$ intervalda $\varphi(x) \equiv 0,\ \psi(x) \equiv 0$ bo‘lsin va barcha $n \in N$ uchun (16) shartlari bajarilsin. Barcha $n \in N$ uchun $\varphi_n \equiv 0,\ \psi_n \equiv 0$ o‘rinli. Natijada, (18) formulaga asosan $y \in [0,T]$ intervalda $u_n(y) = 0$ bo‘ladi va (9) formula bo‘yicha quyidagicha yoziladi:

$$\int_0^l u(x, y) X_n(x)\,dx = 0, \quad n = 1, 2, ... \qquad (20)$$

Demak, $L_2[0,l]$ fazoda $\left\{\sqrt{\frac{2}{l}}\sin\mu_n x\right\}$ sinuslar sistemasi to‘laligiga asosan har qanday $y \in [0,T]$ uchun $[0,l]$ning deyarli hamma joyida $u(x, y) = 0$ bo‘ladi. Chunki (2) shart tufayli $u(x, y)$ funksiya $\overline{D}$ da uzluksiz, u holda $\overline{D}$ da $u(x, y) \equiv 0$. Demak, (2)-(5) masala yagona yehimga ega.

Shuni takidlash kerakki, (10) shartlar u_x hosilaning D to‘rtburchagining yon tomonlariga yaqinlashganda qanday o‘zgarishini belgilaydi. Ya’ni $x \to 0+0$ va $x \to l-0$ − holatlarida u_x hosilasi cheksizlikka intilishi mumkin, lekin (10) shartlari bajarilishi uchun bu holat mumkin.

Quyidagi savollar paydo bo‘ladi: 1) $T, l, b\ va\ n \in N,$ larning qaysi qiymatlarida $\delta(n) = 0;$ 2) agar $\delta(n)$ ifodasining nollari mavjud bo‘lsa, ular nechta va qanday joylashganligi; 3) agar sanoqli sonda nollar to‘plami mavjud bo‘lsa, unda $T, l, b\ va\ n \in N,\ \delta(n) = 0;$ musbat raqamlari va $C_0, n_0 (n_0 \in N)$ doimiylari mavjudmi? barcha $n > n_0$ uchun $\delta(n)$ ifoda mos asimptotikalar bilan noldan ajratiladi. Bu savollarga javob berish uchun $\delta(n)$ ni quyidagi shaklda ifodalaylik

$$\delta(n) = \sin \pi n T \lambda_n \qquad (21)$$

Bu yerda $T = T/l$, $\lambda_n = \sqrt{1+(bl/\pi n)^2}$. (21) ifodaga asosan , T ga nisbatan $\delta(n)=0$ ifoda faqat quyidagi hollarda bajariladi:

$$T = k/n\lambda_n\,,\; k,n \in \mathrm{N} \qquad (22)$$

$M = \{m_{kn} = k/n\lambda_n \quad k,n \in \mathrm{N}\}$ bilan belgilaymiz, T ga nisbatan tenglamaning nollar to‘plami $\delta(n)=0$, ya’ni bazi k va n uchun $T = m_{kn}$ bo‘lsa, $\delta(n)=0$ bo‘ladi. Chunki T ixtiyoriy musbat son, agar T M to‘plamining elementi bo‘lmasa, u $\delta(n)$ ildizlariga qanchalik yaqin bo‘lsa ham, nol qiymat qabul qila olmaydi. Shuning uchun katta n qiymatlarida, bunday T uchun $\delta(n)$ juda kichik bo‘lishi mumkin, ya'ni "kichik maxrajlar" muammosi yuzaga keladi. [13], [15] Shunday qilib, bunday holatni oldini olish uchun, $T > 0$ va musbat C_0 va n_0 doimiylar mavjudligini isbotlaymiz, shunda $\delta(n)$ ifodasi nolga yaqinlashmaydi va uning tegishli asimptotikasi bilan ajratilgan bo‘ladi.

1-lemma. Agar quyidagi shartlardan biri bajarilsa:

1. T istalgan natural son bo‘lsa;
2. $T = p/q$ istalgan ratsional son bo‘lsa, bunda $p,q \in N$, $(p,q)=1$, $p/q \notin N$ u holda C_0 va n_0 $(n_0 \in N)$ musbat doimiy sonlar mavjud bo‘lib, barcha $n > n_0$ uchun quyidagi baholash to‘g‘ri bo‘ladi:

$$|\delta(n)| \geq \frac{C_0}{n} > 0$$

(23)

Isbot. $\tilde{\lambda}_n$ ifodasi bl ga bog‘liq bo‘lib, agar quyidagi shartlar bajarilsa:

$$\frac{bl}{\pi} < 1 \;\; yoki \;\; n > \frac{bl}{\pi} = 1 \qquad (24)$$

U holda ifodani quyidagicha yozish mumkin:

$$\tilde{\lambda}_n = \left(1+\left(\frac{bl}{\pi n}\right)^2\right)^{1/2} = 1+\theta_n,$$

(25)

Bu holda, θ_n ni quyidagicha baholash to‘g‘ri bo‘ladi:

$$\frac{3}{8}\left(\frac{bl}{\pi n}\right)^2 < \theta_n < \frac{1}{2}\left(\frac{bl}{\pi n}\right)^2 \qquad (26)$$

Shunday qilib, (25) ifodani hisobga olgan holda, (21) munosabatni quyidagi ko‘rinishga keltiramiz:

$$\delta(n)=\sin(\pi n\tilde{T}+\tilde{T}\tilde{\theta}_n),\quad \tilde{\theta}_n=\pi n\tilde{\theta}_n$$

(27)

1) Agar, $T=p$ bu yerda, $p\in N$ bo‘lsa, unda (27) ifodasi quyidagi ko‘rinishni oladi:

$$\left|\delta(n)\right|=\left|\sin(\pi n\tilde{T}+\tilde{T}\tilde{\theta}_n)\right|=\left|\sin\left(p\tilde{\theta}_n\right)\right|.$$

(28)

Chunki $\tilde{\theta}_n$ ketma-ketlik (26) baholashga ko‘ra $n\to\infty$da cheksiz kichik qiymat qabul qiladi. Demak $n_2\in N$ son mavjud bo‘lib, barcha $n>n_2$ uchun quyidagi tengsizlik o‘rinli bo‘ladi:

$$0<p\tilde{\theta}_n<\frac{\pi}{2}.$$

Shunda, ma'lum tengsizlikka asosan, quyidagilarni qo‘llashimiz mumkin:

$$\left|\sin x\right|\ge\frac{2}{\pi}\left|x\right|,\quad 0\le\left|x\right|\le\frac{2}{\pi}, \tag{29}$$

(26) va (28) ifodalarni hisobga olgan holda, $n\ge\max\{n_1,n_2\}$ bo‘lganda quyidagi tengsizlikni hosil qilamiz:

$$\left|\delta(n)\right|>\frac{2}{\pi}p\tilde{\theta}_n\ge\frac{2p}{\pi}\frac{3}{8}\frac{(bl)^2}{\pi n}=\frac{1}{n}\frac{3b^2n}{4\pi^2}\ge\frac{\tilde{C}_1}{n}>0.$$

(30)

2) Agar $T={}^{p}\!/_{q}$ $(p,q)=$ bo‘lsa, pn ni q ga qoldiqli bo‘linishini ko‘rib chiqamiz: $np=sq+r,\ \ s,r\in N_0=N\cup\{0\},\ 0\le r<q.$ U holda (27) ifodasini quyidagi ko‘rinishga keltiramiz:

$$\delta(n)=(-1)^s\sin(\frac{\pi r}{q}+\tilde{T}\tilde{\theta}_n) \tag{31}$$

$r=0$ bo‘lgan holat yuqorida ko‘rib chiqilgan. Faraz qilaylik, $r>0$ bo‘lsin. Bu holda, $1\le r\le q-1,\ \ q\ge 2$ tengsizlik hosil bo‘ladi. Chunki $\delta(n)=\sin(\frac{\pi r}{q}+\tilde{T}\tilde{\theta}_n)$ ifoda $n\to\infty$da cheklangan, shuning uchun n_3 kabi biror tabiiy son mavjud bo‘lib, barcha $n>n_3$uchun (31) ifodadan quyidagini tengsizlikni hosil qilamiz:

$$\left|\delta(n)\right|\ge\frac{1}{2}\left|\sin\frac{\pi r}{q}\right|\ge\frac{1}{2}\left|\sin\frac{\pi}{q}\right|\ge\tilde{C}_2\ge\frac{\tilde{C}_2}{n}>0.$$

(32)

Shunday qilib, (30) va (32) tengsizliklardan (23) baholashning to‘g‘riligi kelib chiqadi, bunda barcha $n > n_0 = \max\{n_1, n_2, n_3\}$. Agar T soni kvadratli irrasional son bo‘lsa, (27) munosabatni quyidagicha yozish mumkin:

$$\delta(n) = (-1)^k \sin\left[\pi n\left(\tilde{T} - k/n\right) + \tilde{T}\tilde{\theta}_n\right],$$

(33)

bu yerda k - ixtiyoriy natural son va har qanday $n \in N$ uchun shunday $k \in N$ $[21]$ mavjudki, quyidagilar o‘rinli bo‘ladi:

$$\left|\tilde{T} - \frac{k}{n}\right| < \frac{1}{2n}. \qquad (34)$$

k sonini shunday tanlaymizki, (34) tengsizlikka muvofiq quyidagi tengsizlik bajarilsin:

$$\left|\pi n\left(\tilde{T} - \frac{k}{n}\right)\right| < \frac{\pi}{2}. \qquad (35)$$

Agar $\tilde{T}$ ikkinchi darajali algebraik son, ya’ni kvadratli irrasional son bo‘lsa, Liuvill teoremasiga ko‘ra ([22], 60-bet), $\tilde{T}$ ga bog‘liq bo‘lgan musbat δ soni mavjud bo‘lib, barcha butun n va k lar $n, k > 0$ uchun quyidagi tengsizlik o‘rinli:

$$\left|\tilde{T} - \frac{k}{n}\right| \geq \frac{\delta}{n^2} \qquad (36)$$

(26) baholashga ko‘ra, quyidagi tengsizlikni hosil qilamiz:

$$0 < \tilde{T}\tilde{\theta}_n < \frac{\tilde{T}}{2}\frac{(bl)^2}{\pi n} = \frac{\tilde{C}_3}{n}$$

(37)

bu yerda $\tilde{C}_3$ doimiy o‘zgarmas son bo‘lib quyidagicha aniqlanadi:

$$\tilde{C}_3 = \frac{\tilde{T}(bl)^2}{2\pi} < \frac{\pi}{2}\ . \qquad (38)$$

Hulosa qilishimiz mumkinki, (35) va (37) tengsizliklardan ikki holat kelib chiqishi mumkin:

1) $\pi/2 \leq \pi k\left(\tilde{T} - k/n\right) + \tilde{T}\tilde{\theta} < \pi/2 + \tilde{C}_3 < \pi,$ 2) $-\pi/2 \leq \pi k\left(\tilde{T} - k/n\right) + \tilde{T}\tilde{\theta} < \pi/2.$

1-holat

$$\left|\sin\left[\pi n\left(\tilde{T} - \frac{k}{n}\right) + \tilde{T}\tilde{\theta}\right]\right| \geq \sin\left(\frac{\pi}{2} + \tilde{C}_3\right) = \cos\tilde{C}_3 \geq \frac{C_4}{n}.$$

(39)

2-holat (29) va (36) tengsizliklarni hisobga olib quyidagi ifodani hosil qilamiz:

$$\left|\sin\left[\pi n\left(\tilde{T}-\frac{k}{n}\right)+\tilde{T}\tilde{\theta}\right]\right|>\frac{2}{\pi}\left|\pi n\left(\tilde{T}-\frac{k}{n}\right)+\tilde{T}\tilde{\theta}\right|\geq 2n\left|\tilde{T}-\frac{k}{n}\right|-\left(\tilde{T}\tilde{\theta}_n\right)\frac{2}{\pi}\geq\frac{2\delta}{n}-\frac{2\tilde{C}_3}{\pi n}=\frac{2}{n}\left(\delta-\tilde{C}_3\right). \tag{40}$$

$\tilde{T},l,b\ va\ \delta$ doimiy o‘zgarmas sonlar bo‘lib, quidagi tengsizlikni qanoatlantiradi:

$$\delta-\frac{\tilde{T}}{2}\left(\frac{bl}{\pi}\right)^2>0, \tag{41}$$

Bu tengsizlik $\tilde{T},l,b$ yetarlicha kichik qiymatlar qabul qilganda har doim bajariladi. (41) tengsizlikdan δ ning qiymatini aniqlaymiz. Shartga ko‘ra, $\tilde{T}$ ikkinchi darajali algebraik son. Demak, u butun koeffitsientlarga ega bo‘lgan ikkinchi darajali $f(x)=a_0+a_1x+a_2x^2$, $a_2>0$ ko‘phadining ildizi bo‘ladi va biz $f(x)$ ni quyidagicha yozib olishimiz mumkin:

$$f(x)=(x-\tilde{T})\,f_1(x), \tag{42}$$

Bu yerda, $f_1(x)=a_2(x-\tilde{T})=a_2(x+\tilde{T}+a_1/a_2)=a_2x+a_2\tilde{T}+a_1$, $f_1(\tilde{T})>0$ ya’ni $a_1+2a_2\tilde{T}>0$. Agar $f_1(\tilde{T})\neq 0$ bo‘lsa, $\tilde{T}$ atrofida $f_1(x)>0$ bo‘ladigangan intervallar to‘plami mavjud, masalan $(\tilde{T}-\varepsilon,\tilde{T}+\varepsilon)$, $\varepsilon>0$.

Endi n va k - ixtiyoriy natural sonlar juftligi bo‘lsin; Agar $\left|\tilde{T}-\frac{k}{n}\right|<\varepsilon$ bo‘lsa, $f_1(\frac{k}{n})>0$ tengsizlik o‘rinli bo‘ladi. Demak, (42) tenglikga asosan $x=\frac{n}{k}$ uchun quyidagi ifodani hosil qilamiz:

$$\frac{k}{n}-\tilde{T}=\frac{f\left(k/n\right)}{f_1\left(k/n\right)}=\frac{n^2a_0+a_1nk+a_2k^2}{k^2f_1\left(k/n\right)} \tag{43}$$

(43) ifodada bo‘linmaning surati nolga teng bo‘lmagan butun son bo‘lib, absolyut qiymati kamida birdan kichik bo‘lmaydi. $f_1(x)$ ning $(\tilde{T}-\varepsilon,\tilde{T}+\varepsilon)$ oraliqdagi yuqori chegarasini M bilan belgilab, (43) tenglamadan quyidagi natijani olamiz:

$$\left|\tilde{T}-\frac{k}{n}\right|\geq\frac{1}{n^2M}. \tag{44}$$

Agar $\left|\tilde{T}-\frac{k}{n}\right|\geq\varepsilon$ bo‘lsa va $n\geq 1$ ekanligini hisobga olsak, quyidagi tengsizlikka ega bo‘lamiz:

$$\left|\tilde{T}-\frac{k}{n}\right|\geq\frac{\varepsilon}{n^2}. \tag{45}$$

Shunday qilib, (44) va (45) baholashlaridan (36) baholashni topamiz, bunda $\delta=\min\{\varepsilon,1/M\}$. Endi δ ni $f(x)$ ko‘phadning koeffitsiyentlari orqali ifodalashimiz mumkin. Biz ε ni shunday tanlashimiz kerakki, $f_1(\tilde{T}-\varepsilon)=2a_2\tilde{T}-a_2\varepsilon+a_1\geq 0$ bo‘lsin. Demak ε quyidagi shartni qanoatlantirishi kerak:

$$\varepsilon\leq 2\tilde{T}a_2+\frac{a_1}{a_2}=\frac{\left(2\tilde{T}a_2+a_1\right)}{a_2}.$$

$\varepsilon=\dfrac{\left(2\tilde{T}a_2+a_1\right)}{a_2\tilde{l}}$, $\tilde{l}\geq 1$ belgilash kiritamiz va quidagi tenglikka ega bo‘lamiz:

$$M=f_1(\tilde{T}+\varepsilon)=2a_2\tilde{T}+a_2\varepsilon+a_1=2a_2\tilde{T}+\frac{2a_2\tilde{T}+a_1}{\tilde{l}}+a_1=\frac{\tilde{l}+1}{\tilde{l}}(2a_2\tilde{T}+a_1)$$

$\varepsilon=\dfrac{1}{M}$ shart bilan δ ni topsak, $\tilde{l}$ ga nisbatan quyidagi tenglama hosil bo‘ladi:

$$a_2\tilde{l}^2-(2\tilde{T}a_2+a_1)^2\tilde{l}-(2\tilde{T}a_2+a_1)^2=0.$$

Ushbu tenglamaning yechimi quyidagicha bo‘ladi:

$$\tilde{l}=\frac{\left(2\tilde{T}a_2+a_1\right)^2+(2\tilde{T}a_2+a_1)\sqrt{\left(2\tilde{T}a_2+a_1\right)^2+4a_2}}{2a_2}.$$

(46)

Shunday qilib, δ soni quyidagi formulaga asosan hisoblanadi:

$$\delta=\frac{2}{2\tilde{T}a_2+a_1+\sqrt{\left(2\tilde{T}a_2+a_1\right)^2+4a_2}}.$$

(47)

Agar (46) tenglikning o‘ng tomoni kamida birga teng bo‘lsa,

$$2\tilde{T}a_2+a_1\geq\sqrt{a_2/2} \qquad (48)$$

(48) shart bajarilgan hisoblanadi. Masalan, $a_2=1,\ a_1=0$ deb olaylik, ya’ni $\tilde{T}$ $x^2-d=0$ tenglamasining yechimi bo‘lsin, bunda $d\in N$ va $\sqrt{d}\notin N$. Demak, $\tilde{T}=\sqrt{d}$ va shunda $\tilde{l}>2$ va $\delta=\dfrac{1}{\tilde{T}+\sqrt{\tilde{T}+1}}=\dfrac{1}{\sqrt{d}+\sqrt{d+1}}$. Shunday qilib, (39) va (40) ga asoslanib quyidagi lemmani isbotladik:

2-lemma. Agar $\tilde{T}$ - ikkinchi darajali irrasional algebraik son bo‘lib, $bl<\pi$, (38) va (41) shartlar bajarilsa, δ (47) formula bo‘yicha (48) shart asosida aniqlanadi. Unda $\tilde{T},l,b$ ga bog‘liq musbat doimiy C_0 son mavjud bo‘lib, har qanday $n\in N$ uchun quyidagi baholash o‘rinli bo‘ladi:

$$\left|\delta(n)\right| > C_0/n. \qquad (49)$$

III. Dirixle masalasi yechimi mavjudligi

Agar (23) yoki (49) baholashlar bajarilsa, (8) va (18) tenglamalarining yechimlari asosida (1) tenglama uchun Dirixle masalasining yechimini Furye qatori ko‘rinishida ifodalash mumkin.

$$u(x,y) = \sum_{n=1}^{+\infty} u_n(y) X_n(x).$$

(50)

Quyida biz $\varphi(x)$ va $\phi(x)$ funksiyalari uchun ma’lum shartlar bajarilganda, (50) qatorining yig‘indisi $u(x,y)$, (2) va (3) shartlarini qanoatlantirishini ko‘rsatamiz.

3-lemma. Agar 1-lemma shartlari bajarilsa, $n > n_0$ va $y \in [0,T]$ uchun quyidagi baholashlar o‘rinli bo‘ladi:

$$\left|u_n(y)\right| \le C_1 n\left(\left|\varphi_n\right| + \left|\phi_n\right|\right),$$

(51)

$$\left|u_n''(y)\right| \le C_2 n^3\left(\left|\varphi_n\right| + \left|\phi_n\right|\right),$$

(52)

Bu yerda C_i - musbat o‘zgarmas son.

(51) va (52) baholashlarning to‘g‘riligini bevosita (18) formula va (23) baholash asosida olish mumkin. (49) yordamida quyidagilar isbotlanadi

4-lemma. Agar 2-lemma shartlari bajarilsa, $n \in N$ uchun (51) va (52) baholashlar to‘g‘ri bo‘ladi.

Aytaylik, 1-lemma shartlari bajarilsin. U holda (50) qator va ikkinchi tartibli hosilalar qatorlari yopiq $\overline{D}$ sohada (51) va (52) baholashlar asosida

$$C_3 \sum_{n=n_0+1}^{+\infty} n^3\left(\left|\varphi_n\right| + \left|\phi_n\right|\right) \qquad (53)$$

sonli qator bilan chegaralangan bo‘ladi.

5-lemma. Aytaylik, $\varphi(x), \phi(x) \in C^4[0,l]$ va quidagi sahrtlar bajarilgan:

$$\varphi^i(0) = \varphi^i(l) = 0, \ \psi^i(0) = \psi^i(l) = 0, \quad i = 0,2.$$

Shunda quyidagi ifodalar o‘rinli bo‘ladi:

$$\varphi_n = \frac{\varphi_n^{(4)}}{\mu_n^4}, \qquad \psi_n = \frac{\psi_n^{(4)}}{\mu_n^4}, \qquad (54)$$

bu yerda

$$\varphi_n^{(4)} = \frac{2}{l}\int_0^l \varphi^{(4)}(x)\sin(\mu_n x)dx, \quad \psi_n^{(4)} = \frac{2}{l}\int_0^l \psi^{(4)}(x)\sin(\mu_n x)dx,$$

va

$$\sum_{n=1}^{+\infty}\left|\varphi_n^{(4)}\right| \le \left\|\varphi^{(4)}(x)\right\|_{L^2[0.l]}^2, \qquad \sum_{n=1}^{+\infty}\left|\psi_n^{(4)}\right| \le \left\|\psi^{(4)}(x)\right\|_{L^2[0.l]}^2$$

(55)

Isbot. (14) va (15) formulalardagi integrallarni to‘rt marta bo‘laklab integrallaymiz. Keyin, 5-lemma shartlarini hisobga olgan holda, (54) tengliklarini hosil qilamiz. Lemma shartlariga ko‘ra, $\varphi^4(x)$ *va* $\psi^4(x)$ funklsiyalar $[0,l]$ oraliqda uzluksizdir, shuning uchun Furyee qatorlari nazariyasidan ma’lumki, $\sum_{n=1}^{+\infty}\left|\varphi_n^{(4)}\right|^2$ va $\sum_{n=1}^{+\infty}\left|\psi_n^{(4)}\right|^2$ qatorlar yaqinlashadi va ular uchun (55) Bessel tengsizliklari bajariladi.

Agar 5-lemma shartlari bajarilsa, (53) qator

$$C_4\sum_{n=n_0+1}^{+\infty}\frac{1}{n}\left(\left|\varphi_n^{(4)}\right| + \left|\psi_n^{(4)}\right|\right). \tag{56}$$

(56) sonli qator yordamida baholanadi. (56) qatorining yaqinlashuvchiligiga asoslanib, Veyershtras teoremasiga ko‘ra (50) qator hamda D sohadagi ikkinchi tartibli hosilali qatorlar absolyut va tekis yaqinlashadi.

Aytaylik, 1-lemma uchun $\tilde{T}$ soni ba’zi $n = p = n_1, n_2, \ldots n_m$ $1 \le n_1 < n_2 < \ldots < n_m \le n_0$, $i = \overline{1,m}$ berilgan natural sonlar va $\delta(p) = 0$, (1) tenglama uchun Dirixle masalasining yechimi mavjud bo‘lishi uchun quyidagi shart bajarilishi zarur va yetarli:

$$\varphi_p \cos\lambda_p T = \psi_p, \quad p = n_1, n_2, \ldots n_m$$

(57)

Bu holda (1) tenglama uchun Dirixle masalasining yechimi qator yig‘indisi ko‘rinishida aniqlanadi:

$$u(x,y) = \left(\sum_{n=1}^{n_1-1} + \ldots + \sum_{n=n_{m-1}+1}^{n_m-11} + \sum_{n=n_m+1}^{\infty}\right) u_n(y)X_n(x) + \sum_p A_p u_p(x,y), \tag{58}$$

bu yerda oxirgi yig‘indida p ning qiymatlari $n_1, n_2, \ldots n_m$ ko‘rinishda bo‘ladi, A_p esa ixtiyoriy o‘zgarmas son bo‘lib, $u_p(x,y)$ funksiyalar quyidagi formula orqali aniqlanadi:

$$u_p(x,y) = \left(\varphi_p \cos\lambda_p y + b_p \sin\lambda_p y\right)\sin\mu_p x,$$

bu yerda b_p ixtiyoriy o‘zgarmas son. Agar (58) formulaning o‘ng qismidagi cheklangan yig‘indilarning yuqori chegara pastki chegaradan kichik bo‘lsa, u holda ushbu yig‘indilar nolga teng deb qabul qilinadi. Shu tariqa, yechim yagona ekanligi isbotlandi.

2-teorema. Faraz qilaylik, $\tilde{T}$ 1-lemma shartlarini qanoatlantiradi va $\varphi(x),\psi(x)$ funksiyalari 5-lemma shartlarini bajaradi. Agar $\delta(n)\neq 0,\ n=\overline{1,n_0}$ bo‘lsa, u holda (1) tenglama uchun Dirixle masalasining yagona yechimi mavjud va bu yechim (50) qator orqali aniqlanadi.

Agar $\delta(n)=0,\ n=n_1,n_2,...n_m\leq n_0$ bo‘lsa, u holda (1) tenglama uchun Dirixle masalasi faqat (57) ortogonal shart bajarilganda yechimga ega bo‘ladi va bu yechim (58) qator yig‘indisi ko‘rinishida aniqlanadi.

2-teorema isboti analogik ravishda 4-lemma asosida aniqlanadi.

3-teorema. Faraz qilaylik, $\tilde{T}$ 2-lemma shartlarini qanoatlantirsin va $\varphi(x),\psi(x)$ funksiyalar 5-lemma shartlarini bajarsin. U holda (1) tenglama uchun Dirixle masalasining yagona yechimi mavjud va bu yechim (50) qator orqali aniqlanadi.

Eslatma. Shuni ta’kidlash kerakki, (1) tenglamadagi b koeffitsient muhim rol o‘ynaydi, chunki u (struna tenglamasidan [15] farqli ravishda) Dirixle masalasi uchun yechimning mavjudligi va yagona ekanligini quyidagi $\tilde{T}=T/l$ sonlar sinfi uchun isbotlash imkonini beradi. $\tilde{T}=T/l$: natural sonlar, ratsional sonlar va ikkinchi darajali algebraik irratsional sonlar.

IV. Dirixle masalasi yechimining turg‘unligi

Quyidagi normalarni ko‘rib chiqamiz:

$$\|u(x,y)\|_{L_2[0,l]}=\|u(x,y)\|_{L_2}=\left(\int_0^l |u(x,y)|^2\,dx\right)^{1/2},\ \ 0\leq y\leq T,$$

$$\|u(x,y)\|_{C(\bar{D})}=\max_{\bar{D}}|u(x,y)| \tag{59}$$

4-teorema. Faraz qilaylik, 2 yoki 3-teoremaning shartlari bajarilgan va $\delta(n)\neq 0,\ n=\overline{1,n_0}$ shart o‘rinli bo‘lsin. Demak, (2)–(5)-masalaning yechimini quyidagicha baholash mumkin:

$$\|u(x,y)\|_{L_2}\leq C_5\left(\|\varphi'\|_{L_2}+\|\psi'\|_{L_2}\right), \tag{60}$$

bu yerda C_5,C_6 doimiy sonlar bo‘lib, chegaralangan $\varphi(x),\psi(x)$ funksiyalarga bog‘liq emas.

Isbot. Agar (10) shartdagi $\|u(x,y)\|_{C(\bar{D})}\leq C_6\left(\|\varphi''\|_{C[0,l]}+\|\psi''\|_{C[0,l]}\right)$ funksiyalar $L_2[0,l]$ fazoda ortonormal bo‘lsa, (50) formula va 3-lemmadan quyidagilarga ega bo‘lamiz:

$$\|u(x,y)\|_{L_2}^2=\sum_{n=1}^{\infty}u_n^2(y)\leq C_1^2\sum_{n=1}^{\infty}n^2\left(|\varphi_n|+|\psi_n|\right)^2\leq 2C_1^2\sum_{n=1}^{\infty}(n^2\varphi_n{}^2+n^2\psi_n{}^2) \tag{61}$$

bu yerda C_n - tegishli to‘plamdagi koeffitsientlar. Shu sababli, tizim ortonormal bo‘lganda, har bir koeffitsientning kvadratlari yig‘indisi orqali yechimning normasi aniqlanadi.

5-lemma shartlariga ko‘ra, φ_n va ψ_n koeffitsientlarini quyidagi ko‘rinishda ifodalash mumkin:

$$\varphi_n = \varphi_n^{(1)} / \mu_n, \quad \psi_n = \psi_n^{(1)} / \mu_n$$

bu yerda

$$\varphi_n^{(1)} = \sqrt{\frac{2}{l}} \int_0^l \varphi'(x) \cos \mu_n x dx, \quad \psi_n^{(1)} = \sqrt{\frac{2}{l}} \int_0^l \psi'(x) \cos \mu_n x dx.$$

(61) tengsizlikdan quyidagi tengsizlikni hosil qilamiz:

$$\left\| u(x,y) \right\|_{L_2}^2 \le \frac{2l^2 C_1^2}{\pi^2} \sum_{n=1}^{\infty} \left(\left| \varphi_n^{(1)} \right| + \left| \psi_n^{(1)} \right| \right)^2 \le C_5^2 \left(\left\| \varphi'(x) \right\|_{L_2}^2 + \left\| \psi'(x) \right\|_{L_2}^2 \right).$$

Demak, (59) baholash to‘g‘ri.

Faraz qilaylik, $(x,y) - D$ sohaning ixtiyoriy nuqtasi. U holda, yana (50) formuladan va (51) baholashdan foydalanib, quyidagini hosil qilamiz:

$$\left| u(x,y) \right| \le \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left| u_n(y) \right| \le C_1 \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} n \left(\left| \varphi_n \right| + \left| \psi_n \right| \right). \tag{62}$$

5-lemma shartlariga asosan:

$$\varphi_n = -\varphi_n^{(2)} / \mu_n^2, \quad \psi_n = -\psi_n^{(2)} / \mu_n^2; \tag{63}$$

bu yerda

$$\varphi_n^{(2)} = \int_0^l \varphi''(x) X_n(x) dx, \quad \psi_n^{(2)} = \int_0^l \psi''(x) X_n(x) dx$$

(62) va (63) ifodalarni hisobga olgan holda quyidagini olamiz

$$\left| u(x,y) \right| \le C_1 \left(\frac{l}{\pi} \right)^{3/2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\left| \varphi_n^{(2)} \right| + \left| \psi_n^{(2)} \right| \right). \tag{64}$$

(64) baholash, Koshi-Bunyakovskiy tengsizligi va $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ tenglikni hisobga olgan holda:

$$\left| u(x,y) \right| \le C_1 \left(\frac{l}{\pi} \right)^{3/2} \frac{\pi}{\sqrt{6}} \left[\left(\sum_{n=1}^{\infty} \left| \varphi_n^{(2)} \right|^2 \right)^{1/2} + \left(\sum_{n=1}^{\infty} \left| \psi_n^{(2)} \right|^2 \right)^{1/2} \right] =$$

$$= \frac{C_1 l^{3/2}}{\sqrt{6\pi}} \left(\left\| \varphi''(x) \right\|_{L_2} + \left\| \psi''(x) \right\|_{L_2} \right) \le C_6 \left(\left\| \varphi''(x) \right\|_{C[0,l]} + \left\| \psi''(x) \right\|_{C[0,l]} \right) \tag{65}$$

(65)ni hosil qilamiz. Chunki:

$$\left\|\varphi''(x)\right\|_{L_2} \le \sqrt{l}\max_{0\le x\le l}\left|\varphi''(x)\right| \le \sqrt{l}\left\|\varphi''(x)\right\|_{C[0,l]} \quad \left\|\psi''(x)\right\|_{L_2} \le \sqrt{l}\left\|\psi''(x)\right\|_{C[0,l]}.$$

shu bilan, (65)-dan (60)-baho kelib chiqadi.

FOYDALANILGAN ADABIYOTLAR:

1. Hadamard J. *Equations aux derivees partielles le cas hyperbolique*, L’Enseignement Math. **35** (1), 25–29 (1936).
2. Huber A. *Die erste Randwertaufgabe f ̈ur geschlossene Bereiche bei der Gleichung uxy = f(x, y)*, Monatsh. Math. und Phys. **39**, 79–100 (1932).
3. Mangeron D. *Sopra un problema al contorno per un,equazione differeziale alle derivate parziali di quarto ordine con le caratteristiche realidoppie*, Rend. Accad. sci. fis. mat. Soc. naz. sci. lett. arti Napoli (2), 29–40 (1932).
4. Bourgin P.G., Duffin R. *The Diriclet problem the virbating string equation*, Bull. Amer. Math. Soc. **45** (12), 851–858 (1939).
5. Bourgin P.G. *The Diriclet problem the damped wave equation*, Duke. Math. J. (7), 97–120 (1940).
6. Jonh F. *Diriclet problem for a hyperbolic equation*, Amer. J. Math. **63** (1), 141–154 (1941).
7. Соболев С.Л. *Пример корректной задачи для уравнения колебания струны с данными на всей границе*, ДАН СССР **73** (4), 707–709 (1956).
8. Александрян Р.А. *О задаче Дирихле для уравнения струны и о полноте одной системы функций в круге*, ДАН СССР **73** (5), 869–872 (1950).
9. Вахания Н.Н. *Об одной краевой задаче с данными на всей границе для гиперболической системы, эквивалентной уравнению колебания струны*, ДАН СССР. **116** (6), 906–909 (1957).
10. Березанский Ю.М. *О задаче Дирихле для уравнения колебания струны*, Украинск. матем. журн. **12** (4), 363–372 (1960).
11. Мосолов П.П. *О задаче Дирихле для уравнений в частных производных*, Изв. вузов. Матем., № 3, 213– 218 (1960).
12. АрнольдВ.И. *Малые знаменатели*, Изв. АН СССР. Сер. матем. **25**, 21–86 (1961).
13. АрнольдВ.И. *Малые знаменатели и проблемы устойчивости движения в классической и небесной механике*, УМН **16**. Вып. 6 (114), 91–192 (1963).
14. Капитонов Б.В. *О разрешимости задачи Дирихле для телеграфного уравнения*, Сиб. матем. журн. **17** (2), 273–281 (1976).
15. Сабитов К.Б. *Задача Дирихле для дифференциальных уравнений в частных производных высоких порядков*, Матем. заметки **97** (2), 262–276 (2015).
16. Березанский Ю.М. *Разложение по собственным функциям самосопряженных операторов* (Наук. Думка, Киев, 1965).

17. Мазья В.Г., Шапошникова Т.О. *Жак Адамар — легенда математики* (МЦНМО, М., 2008).
18. АрнольдВ.И. *Математическое понимание природы*. 2 изд. (МЦНМО, М., 2010).
19. Сабитов К.Б. *Задача Дирихле для уравнений смешанного типа в прямоугольной области*, Докл. РАН **413** (1), 23–26 (2007).
20. Моисеев Е.И. *О разрешимости одной нелокальной краевой задачи*, Дифференц. уравнения **37** (11), 1565–1567 (2001).
21. Сабитов К.Б., Сафин Э.М. *Обратная задача для уравнения смешанного параболо-гиперболического типа*, Матем. заметки **87** (6), 907–918 (2010).
22. Хинчин А.Я. *Цепные дроби* (Наука, М., 1978).
23. Бейтмен Г., Эрдейи А. *Высшие трансцендентные функции* (М., 1966).